Why study ISR?

• You get to learn about many useful things, in substantial depth.
  - Plasma physics
  - Radar
  - Coding
  - Electronics (Power, RF, DSP)
  - Signal Processing

• But what if I probably won’t stay in this field?
  - See above!
Outline

• Principle of Pulsed Doppler Radar
• The Doppler spectrum of the ionospheric plasma
• Mathematics of Doppler Processing
• Pulse Compression
The Ubiquitous deciBel

The relative value of two things, measured on a logarithmic scale, is often expressed in deciBel’s (dB)

Example: SNR

\[
\text{Signal-to-noise ratio (dB)} = 10 \log_{10} \left( \frac{\text{Signal Power}}{\text{Noise Power}} \right)
\]

<table>
<thead>
<tr>
<th>Factor of:</th>
<th>Scientific Notation</th>
<th>dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(10^{-1})</td>
<td>-10</td>
</tr>
<tr>
<td>0.5</td>
<td>(10^{0.3})</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>(10^{0})</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(10^{0.3})</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>(10^{1})</td>
<td>10</td>
</tr>
<tr>
<td>100</td>
<td>(10^{2})</td>
<td>20</td>
</tr>
<tr>
<td>1000</td>
<td>(10^{3})</td>
<td>30</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000,000</td>
<td>(10^{6})</td>
<td>60</td>
</tr>
</tbody>
</table>

- 3 dB = 2
- dBW = dB relative to 1 watt
- dBm = dB relative to 1 milliwatt
- dBsm = dB relative to 1 square meter of radar cross section
- dBi = dB relative to isotropic radiation
Waves versus Pulses

What do radars transmit?

Waves?

or Pulses?

Waves, modulated by “on-off” action of pulse envelope

How many cycles are in a typical pulse?

PFISR frequency: 449 MHz
Typical long-pulse length: 480 µs
215,520 cycles!
Pulsed Radar

Power

Peak power

- Pulse length: 100 μsec
- Target Return
- 1 Mega-Watt
- Inter-pulse period (IPP): 1 msec
- Time

Duty cycle = \( \frac{\text{Pulse length}}{\text{Pulse repetition interval}} \) = 10%

Average power = Peak power * Duty cycle = 100 kWatt

Pulse repetition frequency (PRF) = 1/(IPP) = 1 kHz

Continuous wave (CW) radar: Duty cycle = 100% (always on)
Distance = Time

**Range resolution:** Set by pulse length given in units of time, $\tau_p$, or length, $c\tau_p$

\[ \Delta R = \frac{c \tau_p}{2} \]

**Maximum unambiguous range:** Set by Inter-pulse Period (IPP)

\[ IPP = \text{Interpulse period (s)} \]

\[ PRF = \text{pulse repetition frequency} = \frac{1}{IPP} \text{ (Hz)} \]

\[ R_u = \frac{c \cdot IPP}{2} \]
Velocity = Frequency

Transmitted signal: \[ \cos(2\pi f_o t) \]

After return from target: \[ \cos \left(2\pi f_o \left(t + \frac{2R}{c}\right)\right) \]

To measure frequency, we need to observe signal for at least one cycle. So we will need a model of how \( R \) changes with time. Assume constant velocity:

\[ R = R_o + v_o t \]

Substituting:

\[ \cos \left(2\pi f_o \left(f_o \frac{2v_o}{c} t + \frac{2\pi f_o R_o}{c}\right) - f_D\right) \]

By convention, positive Doppler frequency shift (Target and radar closing)
Two key concepts:

- **Distant** ↔ **Time**
  \[ R = \frac{c \Delta t}{2} \]

- **Velocity** ↔ **Frequency**
  \[ v = - \frac{f_D \lambda_0}{2} \]

A Doppler radar measures backscattered power as a function of range and velocity. Velocity is manifested as a Doppler frequency shift in the received signal.
Two key concepts:

- **Distant** ↔ **Time**
  \[ R = c \Delta t / 2 \]

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Two key concepts:

- **Distant ↔ Time**
  \[ R = c\Delta t/2 \]
- **Velocity ↔ Frequency**
  \[ v = -f_D\lambda_0/2 \]

If there is a distribution of targets moving at different velocities (e.g., electrons in the ionosphere) then there is no single Doppler shift but, rather, a Doppler spectrum.

**What is the Doppler spectrum of the ionosphere at UHF (\( \lambda \) of 10 to 30 cm)?**
Longitudinal Modes in a Thermal Plasma

Ion-acoustic

\[ \omega_s = C_s k \quad C_s = \sqrt{\frac{k_B (T_e + 3T_i)}{m_i}} \]

\[ \omega_{si} = -\sqrt{\frac{\pi}{8}} \left[ \left( \frac{m_e}{m_i} \right)^{\frac{1}{2}} + \left( \frac{T_e}{T_i} \right)^{\frac{3}{2}} \exp \left( -\frac{T_e}{2T_i} - \frac{3}{2} \right) \right] \omega_s \]

Langmuir

\[ \omega_L = \sqrt{\omega_{pe}^2 + 3 k^2 v_{the}^2} \approx \omega_{pe} + \frac{3}{2} v_{the} \lambda_D k^2 \]

\[ \omega_{Li} \approx -\sqrt{\frac{\pi}{8}} \frac{\omega_{pe}^3}{k^3 v_{the}^3} \exp \left( -\frac{\omega_{pe}^2}{2k^2 v_{the}^2} - \frac{3}{2} \right) \omega_L \]
Simulated ISR Doppler Spectrum

Particle-in-cell (PIC):
\[
\frac{d \mathbf{v}_i}{dt} = \frac{q_i}{m_i} (\mathbf{E}(\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(\mathbf{x}_i))
\]
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]
\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\]
\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}
\]
\[
\nabla \cdot \mathbf{B} = 0
\]

Simple rules yield complex behavior

![Simulated ISR Doppler Spectrum](image_url)
ISR Measures a Cut Through This Surface

Ion-acoustic “lines” are broadened by Landau damping.
The ISR model

\[
\sigma(\omega) = \left| 1 + \left( \frac{\lambda}{4\pi} \right)^2 \sum_i \left( \frac{1}{D_i} \right)^2 F_i(\omega) \right|^2 \left| N^0_e(\omega) \right|^2 + \left( \frac{\lambda}{4\pi D_e} \right)^4 \left| F_e(\omega) \right|^2 \sum_i \left| N^0_i(\omega) \right|^2
\]

where:

\[
F_e(\omega) = 1 - \omega \int_0^\infty \exp \left( - \frac{16\pi^2 K T_e}{\lambda^2 m_e} \tau^2 \right) \sin(\omega \tau) d\tau
\]

\[
- j \omega \int_0^\infty \exp \left( - \frac{16\pi^2 K T_e}{\lambda^2 m_e} \tau^2 \right) \cos(\omega \tau) d\tau
\]

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\]

From Evans, IEEE Transactions, 1969

Figure 2 · 5: The top figure shows an Incoherent Scattering Spectrum, including the three lines. The middle figure shows a zoom to the ion acoustic line, which is the focus of this research. The bottom figure shows the autocorrelation function $\rho(\tau)$ of the ion acoustic line.

$\frac{f}{f_i}$ is the Doppler frequency associated with the ion acoustic phase velocity.

$\frac{T_e}{T_i}$

area \(\sim N_e^{15}\)
Incoherent Averaging

We are seeking to estimate the power spectrum of a Gaussian random process. This requires that we sample and average many independent “realizations” of the process.

Uncertainties \( \propto \frac{1}{\sqrt{\text{Number of Samples}}} \)
Components of a Pulsed Doppler Radar

- Plasma density ($N_e$)
- Ion temperature ($T_i$)
- Electron temperature ($T_e$)
- Bulk velocity ($V_i$)
Essential Mathematical Operations

Euler:
\[ Ae^{j\phi t} = A \cos(\phi) + jA \sin(\phi) \]
\[ = I + jQ \]

Fourier:
\[ f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} \, dt \quad \iff \quad F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} \, dt \]

Convolution:
\[ f(t) \ast g(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) \, d\tau \quad f(t) \ast g(t) \iff F(\omega) \cdot G(\omega) \]

Correlation:
\[ f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f^*(\tau) \cdot g(t + \tau) \, d\tau \quad f(t) \ast g(t) \iff F(f)^* \cdot G(f) \]

Wiener-Khinchine Theorem:
\[ u(t) \otimes u^*(-t) \iff U(f) \cdot U^*(f) = |U(f)|^2 \]
Dirac Delta Function

\[ \delta(t) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \]

\(\delta(t)\) is defined by the property that for all continuous functions

\[ f(0) = \int_{-\infty}^{+\infty} \delta(t) f(t) dt \]

\[ f(t - T) = f(t) \ast \delta(t - T) \]

The Fourier Transform of a train of delta functions is a train of delta functions.

\[ \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \overset{\mathcal{F}}{\longrightarrow} \quad \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta \left( f - \frac{k}{T} \right) \]
Harmonic Functions

\[ x(t) = 1 \]

\[ X(\omega) = 2\pi\delta(\omega) \]

\[ \cos \omega_0 t \iff \pi [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \]

\[ x(t) \quad \cos \omega_0 t \]

\[ X(\omega) \quad \pi \]

\[ -\omega_0 \quad 0 \quad \omega_0 \quad \omega \]

\[ \sin \omega_0 t \iff j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \]

\[ e^{j\omega_0 t} \iff 2\pi \delta(\omega - \omega_0) \]
Gate function

\[
\text{rect}(t/\tau) = \begin{cases} 
1 & \text{for } -\tau/2 < t < \tau/2 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\text{rect}\left(\frac{t}{\tau}\right) \leftrightarrow \tau \text{ sinc}\left(\frac{\omega \tau}{2}\right)
\]

Not surprisingly, the ISR ACF looks like a sinc function…
The F.T. of a simple RF pulse is a sync function shifted to the carrier frequency (by convolution property and definition of delta function).
The F.T. of a finite train of RF pulses is a decaying train of sync functions with width inversely proportional to the total number of pulses, and separation inversely proportional to the Interpulse Period (IPP).
The F.T. of an infinite train of RF pulses is a line spectrum with lines separated by the pulse repetition frequency.
Bandwidth of a pulsed signal

Spectrum of receiver output has sinc shape, with sidelobes half the width of the central lobe and continuously diminishing in amplitude above and below main lobe

A 1 microsecond pulse has a null-to-null bandwidth of the central lobe = 2 MHz

Two possible bandwidth measures:

“null to null” bandwidth

\[ B_{nn} = \frac{2}{\tau} \]

“3dB” bandwidth

\[ B_{3dB} = \frac{1}{\tau} \]

Unless otherwise specified, assume bandwidth refers to 3 dB bandwidth
Pulse-Bandwidth Connection

Shorter pulse $\iff$ Larger bandwidth
We send a pulse of duration $\tau$. How should we listen for the echo?

- To determine range, we only need to find the rising edge of the pulse we sent. So make $T_1 \ll T_2$.
- But that means large receiver bandwidth, lots of noise power, poor SNR.
- Could make $T_1 \gg T_2$, then we’re integrating noise in time domain.
- So how long should we close the switch?
Detection of a signal embedded in noise

Exponential pulse buried in random noise. Since the signal and noise overlap in both time and frequency domains, the best way to separate them is not obvious.
Most important consideration: Match the bandwidth of the signal you are looking for.

FIGURE 17-8
Example of optimal filters. In (a), three filter kernels are shown, each of which is optimal in some sense. The corresponding frequency responses are shown in (b). The moving average filter is designed to have a rectangular pulse for a filter kernel. In comparison, the filter kernel of the matched filter looks like the signal being detected. The Wiener filter is designed in the frequency domain, based on the relative amounts of signal and noise present at each frequency.
The matched filter is a filter whose impulse response, or transfer function, is determined by a given signal, in a way that will result in the maximum attainable signal-to-noise ratio at the filter output when both the signal and white noise are passed through it.

The optimum bandwidth of the filter, \( B \), turns out to be very nearly equal to the inverse of the transmitted pulse width.

To improve range resolution, we can reduce \( \tau \) (pulse width), but that means increasing the bandwidth of transmitted signal = More noise...
Doppler Revisited

Transmitted signal: \( \cos(2\pi f_o t) \)

After return from target: \( \cos\left[2\pi f_o \left(t + \frac{2R}{c}\right)\right] \)

To measure frequency, we need to observe signal for at least one cycle. So we will need a model of how \( R \) changes with time. Assume constant velocity:

\[
R = R_o + v_o t
\]

Substituting:

\[
\cos\left[2\pi \left(f_o + f_o \frac{2v_o}{c}\right) t + \frac{2\pi f_o R_o}{c}\right] - f_D
\]

By convention, positive Doppler frequency shift \( \leftrightarrow \) Target and radar closing

\[
f_D = \frac{-2f_o v_o}{c} = \frac{-2v_o}{\lambda_o}
\]
Doppler Detection: Intuitive Approach

Phasor diagram is a graphical representation of a sine wave

$I \& Q$ components*

$I \Rightarrow$ in-phase component

\[ A \cos(\phi) \]

$Q \Rightarrow$ in-quadrature component

\[ A \sin(\phi) \]

*relative to reference signal

Consider strobe light as cosine reference wave at same frequency but with initial phase = 0
Doppler Detection: Intuitive Approach

Closing on target – positive Doppler shift

Target’s Doppler frequency shows up as a pulse-to-pulse shift in phase.
I and Q Demodulation

The fundamental output of a pulsed Doppler radar is a time series of complex numbers.

in-phase (I) channel:
\[ p_{rec}(t) \cos(\omega_c t) = a(t) \cos(\phi(t) + \omega_c t) \cos(\omega_c t) \]
\[ = a(t) \frac{1}{2} \left( \cos(\phi(t) + 2\omega_c t) + \cos \phi(t) \right) \]

quadrature (Q) channel (90° out of phase):
\[ p_{rec}(t) \sin(\omega_c t) = a(t) \cos(\phi(t) + \omega_c t) \sin(\omega_c t) \]
\[ = a(t) \frac{1}{2} \left( -\sin(\phi(t) + 2\omega_c t) + \sin \phi(t) \right) \]

I and Q channels together give the analytic signal
\[ s_{rec}(t) = a(t)e^{i\phi(t)} \]

The fundamental output of a pulsed Doppler radar is a time series of complex numbers.
I and Q Demodulation in Frequency Domain

Transmitted signal

\[ \cos(2\pi f_o t) \]

Doppler shifted

\[ \cos(2\pi (f_o + f_D) t) \]
\[ \cos(2\pi f_o t) \]
\[ \cos(2\pi f_D t) \]

Cosine is even function, so sign of \( f_D \) (and, hence, velocity) is lost. What we need instead is:

\[ \exp(j 2\pi f_D t) = \cos(2\pi f_D t) + j \sin(2\pi f_D t) \]

The analytic signal \( \exp(j 2\pi f_D t) \) cannot be measured directly, but the cos and sin components via mixing with two oscillators with same frequency but orthogonal phases. The components are called “in phase” (or I) and “in quadrature” (or Q):

\[ A \exp(j 2\pi f_D t) = I + jQ \]
Example: Doppler Shift of a Meteor Trail

- Collect N samples of $I(t_k)$ and $Q(t_k)$ from a target
- Compute the complex FFT of $I(t_k) + jQ(t_k)$, and find the maximum in the frequency domain
- Or compute “phase slope” in time domain.
Does this strategy work for ISR?

Typical ion-acoustic velocity: 3 km/s
Doppler shift at 450 MHz: 10 kHz
Correlation time: 1/10 kHz = 0.1 ms
Required PRF to probe ionosphere (500 km range): 300 Hz

Plasma has completely decorrelated by the time we send the next pulse.
Does this strategy work for ISR?

Typical ion-acoustic velocity: 3 km/s
Doppler shift at 450 MHz: 10 kHz
Correlation time: 1/10 kHz = 0.1 ms
Required PRF to probe ionosphere (500 km range): 300 Hz

Alternately, the Doppler shift is well beyond the max unambiguous Doppler defined by the Inter-Pulse Period $T$.
The ISR target is “overspread”

\[ f_d \gg 1/\tau \] (Doppler changes significantly during one pulse)

- Must sample multiple times per pulse
- Result: Doppler can be determined from single pulse.
ISR Receiver: Doppler filter bank

Practical Problem: It is hard to make narrow band (High Q) RF filters:

\[ Q = \frac{f_0}{f_H - f_L} \]
ISR Receiver: I and Q plus decorrelation

We have time series of $V(t) = I(t) + jQ(t)$, how do I compute the Doppler spectrum?

Estimate the autocorrelation function (ACF) by computing products of complex voltages ("lag products")

$$\rho(\tau) = \frac{\langle V(t)V^*(t + \tau) \rangle}{S}$$

Power spectrum is Fourier Transform of the ACF

$$\text{FFT}$$
Pulse compression and matched

“If you know what you’re looking for, it’s easier to find.”

Problem: Find the precise location of the target in the image.
Solution: Correlation

a. Image to be searched
b. Target
c. Kernel
Range detection: revisited

\[ \Delta R = \frac{c\tau}{2} = \frac{c}{2B} \]

\( \tau = \text{Pulse length} \)
\( B = \text{Bandwidth} \)

• For high range resolution we want short pulse ⇔ large bandwidth

• For high SNR we want long pulse ⇔ small bandwidth

• Long pulse also uses a lot of the duty cycle, can’t listen as long, affects maximum range

• The Goal of pulse compression is to increase the bandwidth (equivalent to increasing the range resolution) while retaining large pulse energy.
Linear Frequency Modulation (LFM or LFM)

\[ s_1(t) = e^{j2\pi f_0 t} s(t) \]

where

\[ s(t) = \text{Rect} \left( \frac{t}{\tau} \right) e^{j\pi \mu t^2} \]
Matched filter detection of a chirp

Since trailing portions of echo take less time to pass through filter, successive portions tend to bunch up: Amplitude of pulse is increased and width is decreased.

Echoes from closely spaced targets, A and B, are merged but, because of coding, separate in output of filter.
Barker codes

\[
\begin{array}{cccccc}
+ & + & + & + & - & + \\
+ & + & + & + & - & + \\
+ & + & + & + & + & - \\
+ & + & + & + & - & - \\
+ & + & + & + & - & + \\
+ & + & + & + & - & + \\
\end{array}
\]

\[
\begin{array}{c}
\text{correlator output} \\
1 \\
-1+1=0 \\
1-1+1=1 \\
1+1-1-1=0 \\
1+1+1+1+1=5 \\
\end{array}
\]

### TABLE 6.2 All Known Binary Barker Codes

<table>
<thead>
<tr>
<th>Code Length</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>11 or 10</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
</tr>
<tr>
<td>4</td>
<td>1110 or 1101</td>
</tr>
<tr>
<td>5</td>
<td>11101</td>
</tr>
<tr>
<td>7</td>
<td>1110010</td>
</tr>
<tr>
<td>11</td>
<td>11100010010</td>
</tr>
<tr>
<td>13</td>
<td>11111001101001</td>
</tr>
</tbody>
</table>

Interpreting a graph of Barker Codes.
Matched filtering of Barker Code

TIME 0 + + - OUTPUT

TIME 1 + + OUTPUT

TIME 2 + + -

TIME 3 + + -

TIME 4 + + -

TIME 5 + + -

TIME 6 + + -

3 bit code

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